# Math 1512 - Exam 1 Study Guide

Daniel Havens
dwhavens@unm.edu

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## Contents

## Summary and Disclaimer

This is a study guide for the first exam for math 1512 at the University of New Mexico (Calculus I). The exam covers sections 1.4-1.6, 1.8, and 2.1-2.5 of Stewart's Calculus. As such, this study guide is focused on that material. I assume that the student reading this study guide is familiar with the material previously covered in college algebra and trigonometry. If a you feel that you need to review this material, you can send me an email, or take a look at Paul's Online Math notes:

https://tutorial.math.lamar.edu/

If you are not in my class, I cannot guarantee how much these notes will help you. With that said, if your TA or instructor has shared these with you, then you will most likely get some use out of them.

## Methods and Techniques

The focus of this exam is on limits and basic derivatives. For this exam, there are a set of rules that we should know, since they allow us to evaluate limits more easily.

**Limit Laws**

When evaluating a limit, the following limit laws are useful. They all hold if all of the limits are defined. The first is called the sum law:

$$\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

The second is called the product law:

$$\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$$

And if $f(x)$ is continuous, we have the composition law:

$$\lim_{x\to a} f\left(g(x)\right) = f\left(\lim_{x\to a} g(x)\right)$$

Note that the above laws don't hold if any of the limits on the left-hand side are undefined (that is, if they do not exist). Also be careful if one of them is infinity, since if you get $0\cdot\infty$ or $\infty-\infty$, you should try to split the limits up differently.

Finally, there is one important theorem for limits that we should know, since we need it to evaluate some limits:

**The Squeeze Theorem**

If

$$g(x) \leq f(x) \leq h(x),$$

then

$$\lim_{x \to a} g(x) \leq \lim_{x \to a} f(x) \leq \lim_{x \to a} h(x)$$

In practice, you want the limits on the left and right to be easy to find, and for them to go to the same thing, that way the limit in the middle has to be equal to the other two limits. We will work through a few examples of this later.

On the other side of things, we have four main rules for derivatives. Note that these are called "rules" and not "laws", like the versions for limits.

**Addition and Multiplication by Constants**

The only derivative rule that isn't called a "rule" is this one. It states

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

and, if $c$ is some number (say, 2, $-3$, or even $\pi$):

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

The next rule, combined with the above, tells us how to take derivatives of polynomials. This is simply called the "power rule".

**Power Rule**

The power rule says that

$$\frac{d}{dx}x^n = nx^{n-1}$$

For the remaining rules, we will switch to Legrange's notation, as it is easier to read for the last three:

**Product Rule**

The product rule, or Leibniz rule, states that

$$(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$$

**Chain Rule**

The chain rule states that

$$(f(g(x)))' = f'(g(x))g'(x)$$

Finally, we have the quotient rule. Students in my class do not need to have this one memorized. Everybody should be aware of it though, since it is often very useful.

**Quotient Rule**

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - g'(x)f(x)}{(g(x))^2}$$

The final item we have to know is how the derivatives of trigonometric functions work.

**Derivatives of Trigonometric Functions**

We have that

$$\frac{d}{dx}\sin(x) = \cos(x)$$

and

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

Finally, it is good to know how minimum and maximum values relate to derivatives.

**Minima and Maxima with Derivatives**

If $f(x)$ has a minimum or maximum at a point $(x, f(x))$, then $f'(x) = 0$.

# Worked Examples

We will now work through some examples. We start with some limits.

**Example:** Evaluate the limit

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1}$$

To do this, we first factor $x^3 - 1$. Specifically we have $x^3 - 1 = (x-1)(x^2+x+1)$. So,

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}=\lim_{x\to 1}x^2+x+1=1^2+1+1=3.$$

And we will do one more regular limit:

**Example:** Evaluate the limit

$$\lim_{x\to 4}\frac{x^2-6x+8}{x-4}$$

To do this, we factor $x^2-6x+8=(x-4)(x-2)$, so

$$\lim_{x\to 1}\frac{x^2-6x+8}{x-4}=\lim_{x\to 4}\frac{(x-4)(x-2)}{x-4}=\lim_{x\to 1}x-2=4-2=2.$$

Now we will do two examples of squeeze theorem problems. In practice, squeeze theorem problems will be one of two types of problems, and will always include $\sin(x)$ or $\cos(x)$. For these problems, the bound $-1\leq\cos(x)\leq 1$ and $-1\leq\sin(x)\leq 1$ are useful.

**Example:** Using the squeeze theorem, evaluate the limit

$$\lim_{x\to 0}\sin\left(\frac{1}{x}\right)(x^2+x)$$

We begin by noting that $-1\leq\sin\left(\frac{1}{x}\right)\leq 1$, so,

$$-(x^2+x)\leq\sin\left(\frac{1}{x}\right)(x^2+x)\leq x^2+x.$$

And by evaluating the limits at zero, we get that

$$\lim_{x\to 0}-(x^2+x)\leq\lim_{x\to 0}\sin\left(\frac{1}{x}\right)(x^2+x)\leq\lim_{x\to 0}x^2+x.$$

Evaluating the limits on the left and right by plugging in $x=0$ gives us that

$$0\leq\lim_{x\to 0}\sin\left(\frac{1}{x}\right)(x^2+x)\leq 0.$$

So,

$$\sin\left(\frac{1}{x}\right)(x^2+x)=0.$$

These problems are longer than regular limit problems, but the good news is that most of the work is in seeing that $-1\leq\sin(x)\leq 1$ (or $\cos(x)$ instead) rather than in factoring.

Next, we will work a few derivatives. First, let's find the derivative of a polynomial:

**Example:** Find the derivative of $f(x) = x^3 + 2x - 1$.
Finding the derivative, we begin by splitting on addition and then follow up with the power rule. This gives us

$$f'(x) = (x^3)' + 2(x') - (1)' = 3x^2 + 2.$$

Next, let's find the derivative of the product of two functions:

**Example:** Find the derivative of $f(x) = \tan(x)\sqrt{x}$.
We should recall that $(\tan(x))' = \sec^2(x)$, and that $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$. So, from the product rule we get that

$$f'(x) = (\tan(x))'\sqrt{x} + (\sqrt{x})'\tan(x) = \sec^2(x)\sqrt{x} + \frac{\tan(x)}{2\sqrt{x}}.$$

And finally, let's find the derivative of the quotient of two functions:

**Example:** Find the derivative of $\frac{\sin(x)}{x}$.
We proceed by the quotient rule. We have that $(\sin(x))' = \cos(x)$, and that $(x)' = 1$, so,

$$\left(\frac{\sin(x)}{x}\right)' = \frac{(\sin(x))'x - (x)'\sin(x)}{x^2} = \frac{x\cos(x) - \sin(x)}{x^2}.$$

Finally, we discuss minima and maxima.

**Example:** Find the $x$ values where the function $f(x) = x^3 - 9x^2 + 24x + 1$ has its minimum and maximum values.
First, we find the derivative. This gives us that $f'(x) = 3x^2 - 18x + 24$. We can factor this to get

$$f'(x) = 3(x-4)(x-2)$$

So, $f(x)$ has its minimum and maximum values at $x = 4$ and at $x = 2$.

## Practice Problems

These practice problems are separate from the unsolved problems. They should be used to make sure that you are confident with the material, and are of approximately the same level of difficulty as the unsolved questions. They also include worked solutions, unlike the unsolved questions section.

1. Find the derivative of $x^2 + \sqrt{x} + \sqrt[3]{x} + \frac{1}{x}$.
2. Evaluate the limit $\lim_{x \to 0} \sin\left(\frac{\pi}{x^2 - x}\right)\sqrt{x}$.
3. Evaluate the limit $\lim_{x \to -1} \frac{x^2 - 3}{x^3 + x^2 - 3x - 3}$.

If it is requested, I will add more practice problems.

# Practice Problem Solutions

1. Find the derivative of $x^2 + \sqrt{x} + \sqrt[3]{x} + \frac{1}{x}$.

**Solution:** To find the above derivative, we first rewrite the problem in powers of $x$. That is, we should find the derivative of

$$x^2 + x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{-1}.$$

From here, a straightforward application of the power rule, and the fact that derivatives split on addition, gives us that the derivative of the above is just

$$2x + \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{3}x^{-\frac{2}{3}} - x^{-2}.$$

2. Evaluate the limit $\lim_{x\to 0} \sin\left(\frac{\pi}{x^2-x}\right)\sqrt{x}$.

**Solution:** We apply the squeeze theorem. Since sin is always between $-1$ and 1, we have that

$$-1 \leq \sin\left(\frac{\pi}{x^2 - x}\right) \leq 1.$$

So,

$$-\sqrt{x} \leq \sin\left(\frac{\pi}{x^2 - x}\right)\sqrt{x} \leq \sqrt{x},$$

and by taking the limits we get that

$$\lim_{x\to 0} -\sqrt{x} \leq \lim_{x\to 0} \sin\left(\frac{\pi}{x^2 - x}\right)\sqrt{x} \leq \lim_{x\to 0} \sqrt{x}.$$

And since the limits on the left and the right are both zero, we have that

$$0 \leq \lim_{x\to 0} \sin\left(\frac{\pi}{x^2 - x}\right)\sqrt{x} \leq 0.$$

So,

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x^2 - x}\right)\sqrt{x} = 0.$$

3. Evaluate the limit $\lim_{x\to -1} \frac{x^2-3}{x^3+x^2-3x-3}$.

**Solution:** We factor the denominator by grouping. Specifically, we have that

$$x^3 + x^2 - 3x - 3 = x^2(x+1) - 3(x+1) = (x^2 - 3)(x+1).$$

So,

$$\lim_{x\to-1} \frac{x^2-3}{x^3+x^2-3x-3} = \lim_{x\to-1} \frac{x^2-3}{(x^2-3)(x+1)} = \lim_{x\to-1} \frac{1}{x+1}$$

which does not exist, since the limit from the right gives you $\infty$, but the limit from the left gives you $-\infty$. To see this, one may draw the graph.

## Unsolved Questions

Here is a list of 20 unsolved questions which I feel are of similar difficulty to what might be asked of you on an exam.

1. Evaluate the limit
$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 6x + 8}$$
2. Evaluate the limit
$$\lim_{x \to 3} \frac{x^2}{(x-3)^2}$$
3. Evaluate the limit
$$\lim_{x \to 1} \frac{x^2 - x}{x^3 - 2x^2 + x}$$
4. Evaluate the limit
$$\lim_{x \to -\infty} \frac{x^2 + 3x - 1}{5x - 3x^2}$$
5. Evaluate the limit
$$\lim_{x \to \infty} \frac{1 - x^2}{x^3 + 1}$$
6. Evaluate the limit
$$\lim_{x \to 3} \sin\left(\frac{x^2}{x-3}\right)(x^2 - 9)$$
7. Evaluate the limit
$$\lim_{x \to 0} \sqrt{x^5 - x^2} \cos\left(\frac{1}{x}\right)$$
8. Evaluate the limit
$$\lim_{x \to \infty} \frac{\sin(x)}{x}$$
9. Differentiate
$$\tan(x)\sqrt[3]{x^2}$$
10. Differentiate
$$x^4 + \sqrt{x} + x\sin(x)$$
11. Differentiate
$$x^2 - 3x + \frac{1}{x^2}$$
12. Differentiate
$$\frac{1}{x^2+1} - \sqrt{x} + \sec(x)$$

13. Differentiate

$$\frac{\sin(x)}{x} + \frac{\tan(x)}{x^2}$$

14. Differentiate

$$\frac{\cos(x)x^3}{\sqrt{x}+1}$$

15. Differentiate

$$\csc(x) + \cos(x^2) + x^2$$

16. Differentiate

$$\cos(\cos(x)) + \cos(x) - 3$$

17. Differentiate

$$\tan(x^2) - \frac{1}{\sqrt{x}} + \cos(x)$$

18. Differentiate

$$\frac{1}{\sqrt{\cos(x)}} + \tan(\sqrt{x})$$

19. Differentiate

$$\tan(\sqrt[3]{x^4}) + \frac{\sin(x) + x^2}{cos(x^2)}$$

20. Differentiate

$$\cos(x)\sin(x^2)\tan(\sqrt{x})x^{-\frac{10}{7}}$$